MAT 303 - Calculus IV with Applications

Practice Midterm I

Stony Brook University
Fall 2023

[20 pts] **Problem 1.** Solve each of the following initial value problems. Include **all steps** in your solution. No credit is given for only stating the solution.

[5 pts] (i) $y' + 2xy = x, \quad y(0) = -2.$

[5 pts] (ii) $t(t+y)y' = y(t-y), \quad y(1) = 1.$

[5 pts] (iii) $3xy^2y' = 3x^4 + y^3, \quad y(1) = 1.$

[5 pts] (iv) $(x + \arctan(y))dx + \dfrac{x+y}{1+y^2}dy, \quad y(0) = 1.$

[20 pts] **Problem 2.** Consider the differential equation

$$\frac{dy}{dx} = 4x\sqrt{y}.$$

[5 pts] (i) Verify that for each $c > 0$ the function

$$y(x) = \begin{cases} 0, & x^2 \leq c \\ (x^2 - c)^2, & x^2 > c \end{cases}$$

is a solution of the equation.

[5 pts] (ii) Provide another solution to the equation that is not of the above form.

[5 pts] (iii) Consider the initial condition $y(a) = b$. For what values of a and b does the initial value problem have a unique local solution? Provide a detailed explanation.

[5 pts] (iv) Consider the initial condition $y(a) = b$. For what values of a and b does the initial value problem have (A) no solution and (B) infinitely many solutions? Provide brief explanations.

[20 pts] **Problem 3.** Let $x(t)$ denote the population of fish **in hundreds** in a lake at time t (in months). Suppose that the limiting population of the lake is 1000 fish and that the birth rate is 100 fish per 100 fish per month.

[5 pts] (i) Assuming that population is logistic, write a differential equation for $x(t)$.

[5 pts] (ii) Determine the equilibrium points of the model, draw the phase diagram, and classify each of them as stable or unstable. Draw several solutions of the differential equation corresponding to different initial populations.

[5 pts] (iii) Suppose that fish is harvested at a rate of h fish per month. Determine the dependence of the number of critical points on the parameter h and then construct the corresponding bifurcation diagram.

[5 pts] (iv) What is the maximum value of h so that harvesting is viable for this lake? With that rate of harvesting, what will the population of the lake approximately be after a very long period of time?

[10 pts] **Problem 4.** Consider the initial value problem

$$y' = x + y, \quad y(0) = 1.$$

[5 pts] (i) Draw a slope field for the differential equation. On the slope field draw the solution of the initial value problem.

[5 pts] (ii) Use Euler's method with step size 0.5 to approximate $y(1.5)$.